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On the Periodic KdV Equation in Weighted Sobolev Spaces

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Abstract. We prove well-posedness results for the initial value problem of the periodic KdV equation as well as KAM type results in classes of high regularity solutions. More precisely, we consider the problem in weighted Sobolev spaces, which comprise classical Sobolev spaces, Gevrey spaces, and analytic spaces. We show that the initial value problem is well posed in all spaces with subexponential decay of Fourier coefficients, and 'almost well posed' in spaces with exponential decay of Fourier coefficients.

1 Results

We consider the initial value problem for the periodic KdV equation,

$$u_t = -u_{xxx} + 6uu_x, \qquad u\big|_{t=0} = u_0, \tag{1}$$

where all functions are considered to be defined on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

One of the first results in this direction is due to Bona & Smith [5], which is that this problem has a unique, global solution for any initial value in one of the

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standard Sobolev space $\mathcal{H}^m = H^m(\mathbb{T},\mathbb{R})$ with $m \geqslant 2$. That is, for each $u_0 \in \mathcal{H}^m$ there exists a unique continuous curve

$$\varphi:\ \mathbb{R} \to \mathcal{H}^m, \quad t \mapsto \varphi(t,u_0)$$

solving the initial value in the sense defined below. Moreover, taken together they define a continuous flow

$$\mathbb{R}\times\mathcal{H}^m \to \mathcal{H}^m, \quad (t,u_0) \mapsto \varphi(t,u_0).$$

Thus, the initial value problem is globally well-posed on $\mathcal{H}^m$ with $m \geqslant 2$ in the sense of Hadamard: solutions exist for all time, are unique, and depend continuously on their initial values.

Well-posedness and boundedness

Before we proceed we fix some notion. Let $\mathcal{H}^r = H^r(\mathbb{T},\mathbb{R})$ be the usual Sobolev spaces for real $r \geqslant 0$. A continuous curve $\varphi:\ I \to \mathcal{H}^r$, with I an intervall containing 0, is called a *solution* of the initial value problem (1), if it solves (1) in the usual sense of distributions with $\varphi(0) = u_0$. It is called *global*, if it exists for all time, that is, $I = \mathbb{R}$.

We then say that the initial value problem (1) is *globally well-posed in* $\mathcal{H}^r$, if it has a global solution for each initial value in $\mathcal{H}^r$, and the resulting flow

$$\mathbb{R}\times\mathcal{H}^r \to \mathcal{H}^r, \quad (t,u) \mapsto \varphi(t,u)$$

is continuous. A stronger notion is the following. We call (1) *globally uniformly well-posed* in an invariant subspace $\mathcal{H}$ of $\mathcal{H}^r$, if it is globally well-posed, and for every compact time interval J the map

$$\mathcal{H} \to C^0(J,\mathcal{H}), \quad u \mapsto \varphi(\cdot,u)$$

is uniformly continuous on bounded subsets of $\mathcal{H}$ with respect to the usual sup-norm on the second space.

Well-posedness in the weighted Sobolev spaces $\mathcal{H}^w$ introduced later is defined analogously.

As it turns out, we actually have to restrict ourselves to invariant subspaces of functions of constant mean value

$$[u] := \int_{\mathbb{T}} u \,\mathrm{d}x,$$

which is invariant under the KdV flow. That is, we have to consider the subspaces

$$\mathcal{H}^w_c = \{u \in \mathcal{H}^w : [u] = c\}, \qquad c \in \mathbb{R}.$$

Such a restriction is *necessary* for the following statements to be correct – see the remark following the supplement to Theorem 1.

Another interesting question in the study of initial value problems such as (1) is the long time behaviour of solutions – in particular, whether they stay bounded or even uniformly bounded for all time. Here we say that solutions of (1) are *uniformly bounded* in a weighted Sobolev space $\mathcal{H}^w$, if they are uniformly bounded within this space for *all time*.

Known results

Since the first results of Temam [35], Sjöberg [34] and Bona & Smith [5], the initial value problem for KdV and its well-posedness have been studied intensively. An excellent overview with a detailed bibliography is provided by the web site created by Colliander, Keel, Staffilani, Takaoka & Tao [12].

One focus has been on *low regularity* solutions, that is, solutions in the Sobolev spaces $\mathcal{H}^r$ with real $r \leqslant 0$. We just mention the works of Bourgain [6, 7, 8, 9], Kenig, Ponce & Vega [24, 25], Colliander, Keel, Staffilani, Takaoka & Tao [11], and Kappeler & Topalov [23]. As a result, KdV is now known to be globally well-posed in $\mathcal{H}^r$ for $r \geqslant -1$, and globally uniformly well-posed in $\mathcal{H}^r_c$ for $r \geqslant -1/2$. See [23] and [11], respectively. Incidentally, it is an interesting phenomenon, that an equation can be globally well-posed, but not in a uniform way.

In this paper we focus on *high regularity* solutions. These are solutions in a general class of weighted Sobolev spaces within $\mathcal{H}^0$, that encompass analytic and Gevrey spaces, among others, as well as the spaces $\mathcal{H}^m$. Some results in this direction on the *real line* are due to Bona, Grujić & Kalisch [4, 18], for example. But in general, the questions of existence and well-posedness of solutions of nonlinear pdes of high regularity have not been widely considered. We argue that this is a topic which deserves to be studied in more depth, revealing important features of the nonlinear equation considered.

Weighted Sobolev spaces

To state our results, we introduce *weighted Sobolev spaces* $\mathcal{H}^w$ within

$$\mathcal{H}^0 = L^2(\mathbb{T},\mathbb{C}) = \left\{u = \textstyle\sum_{n\in\mathbb{Z}} u_n \mathrm{e}^{2\pi i n x} : \sum_{n\in\mathbb{Z}} |u_n|^2 < \infty\right\}$$

as follows [20, 21]. First, a *weight* is any function $w\colon \mathbb{Z} \to \mathbb{R}$, which is *normalized*, *symmetric*, and *submultiplicative*:

$$w_n \geqslant 1, \qquad w_{-n} = w_n, \qquad w_{n+m} \leqslant w_n w_m$$

for all n and m. The w-norm of a function u in $\mathcal{H}^0$ is then defined through

$$\|u\|_w^2 := \sum_{n\in\mathbb{Z}} w_n^2 \, |u_n|^2 ,$$

and

$$\mathcal{H}^w := \left\{ u \in \mathcal{H}^0 : \|u\|_w < \infty \right\}$$

is the Banach space of all such functions with finite w-norm. Note that $\mathcal{H}^0 = \mathcal{H}^w$ for the trivial weight $w \equiv 1$.

Here are some examples of relevant weights. Let $\langle n\rangle = 1 + |n|$.

The *Sobolev weights*

$$\langle n\rangle^r, \qquad r \geqslant 0,$$

give rise to the usual Sobolev spaces $\mathcal{H}^r$ of 1-periodic, real-valued functions. In particular, for nonnegative integers m we obtain the standard spaces $\mathcal{H}^m$.

The *Abel weights*[1]

$$\langle n\rangle^r \mathrm{e}^{a|n|}, \qquad r \geqslant 0, \ \ a > 0,$$

define spaces $\mathcal{H}^{r,a}$ of functions in $\mathcal{H}^r$ referred to as *Abel spaces*, which are analytic on the complex strip $|\mathrm{Im}\, z| < a/2\pi$ and have traces in $\mathcal{H}^r$ on the boundary lines.

The *Gevrey weights*

$$\langle n\rangle^r \mathrm{e}^{a|n|^\sigma}, \qquad r \geqslant 0, \ \ a > 0, \ \ 0 < \sigma < 1,$$

lie in between and give rise to the so called *Gevrey spaces* $\mathcal{H}^{r,a,\sigma}$. They are all subspaces of $C^\infty(\mathbb{T},\mathbb{R})$. Obviously,

$$\mathcal{H}^{r,a} = \mathcal{H}^{r,a,1} \subsetneq \mathcal{H}^{r,a,\sigma} \subsetneq \mathcal{H}^{r,a,0} = \mathcal{H}^r,$$

so the Gevrey spaces interpolate between the Abel and Sobolev spaces for the same r and a. The latter are obtained for $\sigma = 1$ and $\sigma = 0$, respectively.

[1] The term *Abel* weights is chosen to go along with *Sobolev* and *Gevrey* weights.

Since $\log w_n$ is subadditive and nonnegative, the limit

$$\chi(w) := \lim_{n\to\infty} \frac{\log w_n}{n}$$

exists and is nonnegative [33, no. 98]. Naturally, we call a weight w *exponential*, if

$$\chi(w) > 0.$$

We call w *subexponential*, if $\chi(w) = 0$ *and in addition* $\log w_n/n$ converges to zero in an eventually monotone manner. This is not a precise dichotomy, but we are not aware of any interesting weight that does not belong to either class.

Clearly, Abel weights are exponential, while Sobolev and Gevrey weights are subexponential. Yet another example of a subexponential weight is given by

$$\langle n\rangle^r \exp\left(\frac{a\,|n|}{1+\log^\alpha\langle n\rangle}\right), \qquad r \geqslant 0, \ \ a > 0, \ \ \alpha > 0,$$

which is lighter than the Abel and heavier than the Gevrey weights.

Our results with respect to well-posedness are optimal for subexponential weights with respect to regularity. In the exponential case, however, due to our method we have to allow for a slight loss of smoothness. Using other techniques such as a priori estimates this can probably be avoided.

Statements of the main theorems

Recall that $\mathcal{H}_c^w = \{u \in \mathcal{H}^w : [u] = c\}$. We first consider the case of *subexponential* weights, where our results are optimal as far as the regularity properties are concerned.

Theorem 1 *The periodic* KdV *equation is globally uniformly well-posed in every space $\mathcal{H}_c^w$ with a subexponential weight w and a real c. That is, for each initial value u in one of these spaces $\mathcal{H}_c^w$, the associated Cauchy problem has a global solution $t \mapsto \varphi^t(u)$ in $\mathcal{H}_c^w$, giving rise to a continuous flow*

$$\mathbb{R}\times\mathcal{H}_c^w \to \mathcal{H}_c^w, \qquad (t,u) \mapsto \varphi^t(u), \tag{2}$$

which is even uniformly continuous on bounded subsets of $\mathbb{R}\times\mathcal{H}_c^w$. Moreover, for bounded subsets of initial values all solution curves are uniformly bounded.

As our results are based on a fairly precise knowledge of the entire KdV flow, we obtain the following additional results almost for free. Recall that a continuous curve $\gamma\colon \mathbb{R} \to \mathcal{H}^w$ is called *almost-periodic*, if for any $\varepsilon > 0$ there exists an $l > 0$ such that any open interval of length l contains an ε-approximate period T:

$$\sup_t \|\gamma(t+T) - \gamma(t)\|_w < \varepsilon.$$

Supplement to Theorem 1 *The flow (2) has the following properties.*

(i) *Each solution is almost-periodic in time.*

(ii) *The Lyapunov exponents of any solution are zero.*

(iii) *Each time shift map φ^t is real analytic as a map $\mathcal{H}_c^w \to \mathcal{H}_c^w$.*

It is worth noting that solutions are *not analytic in t*. As their frequencies are unbounded, they do not even have a proper tangent vector in the t-direction. For the same reason, they are also not analytic with respect to the mean value c. If c is complex, then the imaginary parts of the frequencies form an unbounded sequence – see for example Appendix A in [23].

In the case of *exponential* weights, our method of proof entails a slight loss of smoothness. On the other hand, it assures that within these spaces of slightly less regular functions the solution curves remain bounded for all time.

Theorem 2 *The periodic* KdV *equation is "almost" globally well-posed in every space $\mathcal{H}_c^w$ with an exponential weight w and real c. That is, for each bounded subset $\mathcal{B}$ of $\mathcal{H}_c^w$ there exists $0 < \rho \leqslant 1$ such that the Cauchy problem for each initial value $u \in \mathcal{B}$ has a global solution $t \mapsto \varphi^t(u)$ in $\mathcal{H}_c^{w^\rho}$. These solutions are uniformly bounded and give rise to a continuous flow*

$$\mathbb{R} \times \mathcal{B} \to \mathcal{H}_c^{w^\rho}, \quad (t,u) \mapsto \varphi^t(u).$$

The Supplement also applies to this case, with each time shift map φ^t being analytic as a map $\mathcal{H}_c^w \to \mathcal{H}_c^{w^\rho}$.

Here, w^ρ is the weight with $(w^\rho)_n = w_n^\rho$, which is again normalized, symmetric and submultiplicative. Thus, for initial values u in a bounded subset $\mathcal{B}$ of $\mathcal{H}_c^{0,a}$, say, (1) has a global solution in $\mathcal{H}_c^{0,\rho a}$ with a fixed $0 < \rho \leqslant 1$. It is an open question, whether ρ can be chosen to be 1. For related results, see for example [1].

Our results are not restricted to the standard KdV equation, but apply simultaneously to all equations in the KdV hierarchy, as defined for instance in [22]. The second KdV equation, for example, reads

$$u_t = u_{xxxxx} - 10uu_{xxx} - 20u_xu_{xx} + 30u^2u_x.$$

Such a hierarchy may be defined in a variety of ways, but this is immaterial here and does not affect the statement of the following theorem.

Theorem 3 *Theorems 1 and 2 and their supplements also hold for every* KdV *equation in the* KdV *hierarchy, provided that in the case of Sobolev spaces $\mathcal{H}_c^r$, r is sufficiently large.*

Using the preceding results we may extend the KAM theory of Hamiltonian perturbations of KdV equations developed by Kuksin [26, 27, 28] and expounded in [29, 22]. Consider the perturbed KdV equation

$$\frac{\partial u}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial H}{\partial u} + \varepsilon\frac{\partial K}{\partial u}\right).$$

If K is real analytic in u with a gradient $\partial K/\partial u$ in some standard Sobolev space $\mathcal{H}_0^m$, $m \geqslant 1$, then KAM for KdV asserts the persistence of *quasi-periodic* solutions for sufficiently small $\varepsilon \neq 0$. This result may now be extended as follows.

Theorem 4 *Under sufficiently small Hamiltonian perturbations, the majority of the quasi-periodic solutions of the* KdV *equation persists, their regularity being the same as the regularity of the perturbing term in the subexponential case, or only slightly less in the exponential case.*

A more detailed statement of this theorem is given in section 4. Incidentally, it answers a question raised by Jürgen Moser many years ago, and which was the main motivation behind this work.

Outline of proof

Our theorems are based on two observations. First, the periodic KdV equation is well known to be an *infinite dimensional, integrable Hamiltonian system*. As such, it even admits global Birkhoff coordinates $(x_n, y_n)_{n\geqslant 1}$ defined as the cartesian counterpart to global action angle coordinates $(I_n, \theta_n)_{n\geqslant 1}$. In these coordinates, the KdV Hamiltonian takes the infinite dimensional classical form

$$H = H(I_1, I_2, \ldots), \qquad 2I_n = x_n^2 + y_n^2,$$

with classical equations of motion. They are trivial to integrate, making the well-posedness problem completely transparent in the underlying sequence spaces.

The so called *Birkhoff map* $u \mapsto (x_n, y_n)_{n\geqslant 1}$ defines a canonical diffeomorphism

$$\Omega\colon\ \mathcal{H}_0^0 \to \mathcal{h}_\star^0$$

between

$$\mathcal{H}_0^0 = \left\{ u \in L_0^2 : \textstyle\int_{\mathbb{T}} u \,\mathrm{d}x = 0 \right\},$$

and a space $\mathfrak{h}_\star^0$ of weighted ℓ^2-sequences defined in (4) below. The second observation is that this map Ω has many features of the Fourier transform. Indeed, the differential of Ω at the origin *is* a weighted Fourier transform. More importantly, one can prove Paley-Wiener type theorems for Ω. For example, for any subexponential weight w, the restriction of Ω to the weighted Sobolev space $\mathcal{H}_0^w$ gives rise to a diffeomorphism

$$\Omega : \quad \mathcal{H}_0^w \to \mathfrak{h}_\star^w .$$

These results rely on a precise correspondence between the regularity properties of a function $u \in \mathcal{H}_0^0$ and the decay properties of its actions I_n. These two are linked by the spectral properties of the associated Hill operator used in the Lax-pair formulation of KdV as follows.

For a *potential* $u \in \mathcal{H}_0^0$ consider the Hill operator

$$L_u = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u$$

on the interval $[0, 2]$ with periodic boundary conditions. Its spectrum, $\operatorname{spec}(u)$, is pure point and consists of an unbounded sequence of periodic eigenvalues

$$\lambda_0(u) < \lambda_1(u) \leqslant \lambda_2(u) < \lambda_3(u) \leqslant \dots ,$$

where the open intervals $(\lambda_{2n-1}(u), \lambda_n(u))$ are refered to as the *gaps* of $\operatorname{spec}(u)$. Lax observed, by way of his famous Lax pair argument, that the KdV flow defines an *isospectral deformation* on the space of potentials u in $\mathcal{H}_0^m$ with $m \geqslant 3$ arbitrary. That is,

$$\operatorname{spec}(\varphi^t(u)) = \operatorname{spec}(u)$$

for any solution $t \mapsto \varphi^t(u)$ of the KdV equation in $\mathcal{H}_0^m$.

The same applies then to the sequence $\gamma(u) = (\gamma_n(u))_{n \geqslant 1}$ consisting of the so called *gap lengths* $\gamma_n(u) = \lambda_{2n}(u) - \lambda_{2n-1}(u)$. So we also have

$$\gamma(\varphi^t(u)) = \gamma(u).$$

It now turns out that on one hand,

$$\gamma_n(u) \sim n I_n(u)$$

uniformly on bounded subsets of $\mathcal{H}_0^m$. Hence, decay properties of the gap lengths translate into decay properties of the actions in a precise way.

On the other hand, the decay properties of $\gamma(u)$ are closely tied to the regularity of u. For example, a classical result due to Marčenko & Ostrowskiĭ [30] states that for any $u \in L_0^2$,

$$u \in \mathcal{H}_0^m \quad \Leftrightarrow \quad \sum_{n \geqslant 1} n^{2m} \gamma_n^{2m}(u) < \infty$$

for any integer $m \geqslant 0$. Characterizations of this type have been studied extensively and extended to a variety of subspaces $\mathcal{H}_0^w$ of $\mathcal{H}_0^0$ for example by Kappeler & Mityagin [21] and Djakov & Mityagin [13]. Pöschel [31] finally shows, by way of a new functional analytic approach, that u belongs to a space $\mathcal{H}_0^w$ with *any* subexponential weight w if and only if $\gamma(u)$ belongs to an analogously defined sequence space ℓ^w defined below. We may thus traverse the implications

$$\begin{array}{ccc} u \in \mathcal{H}_0^w & \Longrightarrow & \gamma(u) \in \mathfrak{h}_\star^w \\ & & \Big\Downarrow \\ \varphi^t(u) \in \mathcal{H}_0^w & \Longleftarrow & \gamma(\varphi^t(u)) \in \mathfrak{h}_\star^w \end{array}$$

clockwise from top left to bottom left and arrive at the conclusion that a solution curve with initial value in $\mathcal{H}_0^w$ stays in $\mathcal{H}_0^w$.

In the exponential case, however, this last argument is no longer true, and we have to allow for some loss in "exponentiality". It is quite likely that this is an artefact of our methods.

The preceding discussion was restricted to functions of mean value zero. The general case, however, is easily reduced to this case. Letting $u = v + c$ with $[v] = 0$ and $c = [u]$ the KdV Hamiltonian introduced in the next section takes the form

$$H(u) = H_c(v) + c^3$$

with

$$H_c(v) = \int_{\mathbb{T}} \left(\tfrac{1}{2} v_x^2 + v^3 \right) \mathrm{d}x + 6c \int_{\mathbb{T}} \tfrac{1}{2} v^2 \,\mathrm{d}x. \tag{3}$$

The additive constant is irrelevant, and the second integral only amounts to a shift in the frequencies of the KdV flow, which is also immaterial to our results. Therefore, it suffices to consider the case $c = 0$ to establish our results. Then we write again H instead of H_0.

The rest of the paper is organized as follows. In section 2 we describe the Birkhoff coordinates for KdV which are constructed in [22], and formulate two addenda, which will imply Theorem 1 and 2. In section 3 we give a precise description of the relationship between the regularity of a potential and its spectral asymptotics, which will imply the addenda of section 2. The proofs of the statements about this relationship, however, are somewhat lengthy and given in a separate paper [31]. In the concluding section 4 we briefly address Hamiltonian perturbations of KdV.

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2 Birkhoff Coordinates

As is well known, the KdV equation can be written as an infinite dimensional Hamiltonian system

$$\frac{\partial u}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial H}{\partial u}.$$

The Hamiltonian is

$$H(u) = \int_{\mathbb{T}} \left(\tfrac{1}{2}u_x^2 + u^3\right)\mathrm{d}x,$$

and as the underlying phase space one may take one of the usual Sobolev spaces

$$\mathcal{H}^m = H^m(\mathbb{T},\mathbb{R})$$

of real valued functions on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, where $m \geqslant 1$ is an integer. The Poisson bracket proposed by Gardner,

$$\{F,G\} = \int_{\mathbb{T}} \frac{\partial F}{\partial u}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial G}{\partial u}\,\mathrm{d}x,$$

makes $\mathcal{H}^m$ a Poisson manifold, on which the KdV equation may also be represented in the form

$$u_t = \{u,H\},$$

familiar from classical mechanics.

The Poisson structure $\{\cdot,\cdot\}$ is degenerate, as it admits the mean value $[\cdot]$ as a *Casimir function*. That is, $[\cdot]$ commutes with *every* other function on $\mathcal{H}^w$. From now on we therefore restrict ourselves to the invariant subspaces $\mathcal{H}_0^w$, where the Poisson structure is nondegenerate and gives rise to a symplectic structure as well.

Next, we introduce the weighted sequence spaces

$$h^w = \ell^w \times \ell^w$$

with elements (x,y), where

$$\ell^w = \left\{x = (x_n)_{n\geqslant 1} : \|x\|_w^2 = \textstyle\sum_{n\geqslant 1} w_n^2\,|x_n|^2 < \infty\right\}.$$

We endow h^w with the standard Poisson structure, for which $\{x_n, y_m\} = \delta_{nm}$, while all other brackets vanish. To simplify notations, we further introduce

$$h_\star^w = \ell_\star^w \times \ell_\star^w, \qquad \ell_\star^w = \left\{x \in \ell^w : (\sqrt{n}x_n)_{n\geqslant 1} \in \ell^w\right\}. \tag{4}$$

The extra weight $\sqrt{n}$ reflects the effect of the derivative $\mathrm{d}/\mathrm{d}x$ in the Gardner bracket.

The following theorem was first proven in [2, 3]. A quite different approach was first presented in [19], and a comprehensive exposition is given in [22].

Theorem 5 *There exists a diffeomorphism $\Omega\colon \mathcal{H}_0^0 \to h_\star^0$ with the following properties.*

(i) *Ω is onto, bi-analytic and bounded, and it takes the standard Poisson bracket into the Gardner bracket.*

(ii) *The restriction of Ω to $\mathcal{H}_0^m$, $m \geqslant 1$, gives rise to a map $\Omega\colon \mathcal{H}_0^m \to h_\star^m$, which is again onto and bi-analytic.*

(iii) *Ω introduces global Birkhoff coordinates for the* KdV *Hamiltonian on $\mathcal{H}_0^1$. That is, on $h_\star^1$ the transformed* KdV *Hamiltonian $H \circ \Omega^{-1}$ is a real analytic function of*

$$I_n = \tfrac{1}{2}(x_n^2 + y_n^2), \qquad n \geqslant 1.$$

(iv) *The last statement also applies to every other Hamiltonian in the* KdV *hierarchy, if '1' is replaced by 'm' with m sufficiently large.*

The construction of Birkhoff coordinates for a function u in $\mathcal{H}_0^0$ actually starts out with the definition of the *actions* I_n and the associated *angles* θ_n. Those are defined in terms of certain path integrals on the two-sheeted Riemann surface of infinite genus associated with the periodic spectrum $\operatorname{spec}(u)$ of u. No reference to the KdV equation is required for this construction. Therefore, the I_n are defined on all of $\mathcal{H}_0^0$, while each θ_n is defined on the dense open subset where $\lambda_{2n} \neq \lambda_{2n-1}$. For all the details we refer to [22].

Denoting the transformed KdV Hamiltonian by the same symbol we thus obtain a real analytic Hamiltonian

$$H = H(I_1, I_2, \dots)$$

on $h^1_\star$. Its equations of motion are the classical ones,

$$\dot{x}_n = H_{y_n}, \quad \dot{y}_n = -H_{x_n}, \qquad n \geqslant 1,$$

since the Poisson structure on $h^1_\star$ is the standard one. It is therefore evident, that every solution of the KdV equation exists for all time, and is indeed *almost periodic*. More precisely, every solution winds around some underlying invariant torus

$$T_I = \prod_{n\geqslant 1} S_{I_n}, \qquad S_{I_n} = \left\{ (x_n, y_n) \in \mathbb{R}^2 : x_n^2 + y_n^2 = 2I_n \right\},$$

which is fixed by the actions of the initial positions. The speed on the n-th circle S_{I_n} is determined by the n-th frequency

$$\omega_n = H_{I_n}(I_1, I_2, \dots),$$

and the entire flow is given by

$$\psi^t(x, y) = (x_n \cos \omega_n t, \, y_n \sin \omega_n t)_{n\geqslant 1}.$$

Obviously, ψ^t preserves all weighted norms and thus all weighted spaces $h^w_\star$.

To obtain our results about the well-posedness of the KdV equation, we now formulate two extensions of item (ii) of Theorem 5, to be proven in the next section. First we consider subexponential weights.

Addendum 1 to Theorem 5 *For each subexponential weight w, the restriction of Ω to $\mathcal{H}_0^w$ gives rise to an onto, bi-analytic and bounded diffeomorphism*

$$\Omega : \ \mathcal{H}_0^w \to h^w_\star.$$

Proof of Theorem 1 and its Supplement. Due to its symplectic nature, Ω maps solution curves $t \mapsto \varphi^t(u)$ in function space into solution curves $t \mapsto \psi^t(x, y)$ in sequence space, with $(x, y) = \Omega(u)$. Since Ω is also a bounded diffeomorphism between $\mathcal{H}_0^w$ and $h^w_\star$, and ψ^t preserves $h^w_\star$, the diagram

$$\begin{tikzcd}
u \in \mathcal{H}_0^w \arrow[r, "\Omega"] \arrow[d, "\varphi^t"'] & (x, y) \in h^w_\star \arrow[d, "\psi^t"] \\
\varphi^t(u) \in \mathcal{H}_0^w & \psi^t(x, y) \in h^w_\star \arrow[l, "\Omega^{-1}"']
\end{tikzcd}$$

commutes and proves the theorem.

The statements of the Supplement also follow by looking at the flow in sequence space. For instance, all solution curves starting in a bounded subset stay uniformly bounded for all time. And the distance of two solution curves grows at most linearly with time, whence their Lyapunov exponents are zero – see also [23]. ∎

The preceding reasoning applies in particular to the Sobolev spaces $\mathcal{H}_0^m$. Thus the well-posedness of KdV in $\mathcal{H}_0^m$ follows directly with Theorem 5, and Addendum 1 is not needed. – Now we consider exponential weights.

Addendum 2 to Theorem 5 *Let w be an exponential weight. Then for every bounded subset B of $h^w_\star$ there exists $0 < \rho \leqslant 1$ such that $\Omega^{-1}(B)$ is a bounded subset of $\mathcal{H}_0^{w^\rho}$.*

Proof of Theorem 2. Let w be an exponential weight, and $\mathcal{B}$ a bounded subset of $\mathcal{H}_0^w$. Then $B = \Omega(\mathcal{B})$ is a bounded subset of $h^w_\star$ by Propositions 1 and 3 below. As the flow ψ^t preserves the $h^w_\star$-norm, the set

$$B^- = \bigcup_{t\in\mathbb{R}} \psi^t(B)$$

is contained in the same centered ball as B. Hence, by the second addendum there exists a $0 < \rho \leqslant 1$ such that $\mathcal{B}^- = \Omega^{-1}(B^-)$ is contained in $\mathcal{H}_0^{w^\rho}$. We obtain the commutative diagram

$$\begin{tikzcd}
\mathcal{B} \subset \mathcal{H}_0^w \arrow[r, "\Omega"] \arrow[d, "\varphi^t"'] & B \subset h^w_\star \arrow[d, "\psi^t"] \\
\mathcal{B}^- \subset \mathcal{H}_0^{w^\rho} & B^- \subset h^w_\star \arrow[l, "\Omega^{-1}"']
\end{tikzcd}$$

which proves the theorem. ∎

Proof of Theorem 3. The proofs of Theorem 1 and 2 and their supplements are based on the fact that the map Ω trivializes the KdV flow in the Birkhoff coordinates. By item (iv) of Theorem 5, however, Ω simultaneously trivializes any other KdV flow in the KdV hierarchy. The only difference is in the frequencies ω_n associated with the circles S_{I_n}, and in the minimal regularity required for the KdV hamiltonians to make sense. Hence the preceding proofs apply to higher KdV equations as well. ∎

3 Regularity

The proofs of the addenda are based on two observations. First, the asymptotics of the Birkhoff coordinates of a function u in $\mathcal{H}_0^0$ are closely related to the asymptotics of its spectral gaps. Second, these asymptotics are very closely related to the regularity of u. The former relation is established in [22], and the latter in [13, 31]. Here, we will quote the relevant results and apply them to the map Ω.

Since the periodic spectrum of u plays a central role, we will often refer to u as a *potential*. The KdV equation, on the other hand, is irrelevant here and not mentioned at all. We also write L_0^2 instead of $\mathcal{H}_0^0$.

In the following we also have to mention potentials in a small complex neighbourhood of L_0^2. We forego a detailed extension of the concept of eigenvalues and gaps to this situation and instead refer for example to [22].

Proposition 1 ([22, p. 67]) *There exists a complex neighbourhood W of L_0^2 such that each quotient I_n/γ_n^2 extends analytically to W and satisfies*

$$8\pi n \frac{I_n}{\gamma_n^2} = 1 + O\left(\frac{\log n}{n}\right), \qquad n \geqslant 1,$$

locally uniformly on W, as well as uniformly on bounded subsets of L_0^2.

This proposition is proven in [22] except for the very last statement. But that follows from the explicit representation of I_n/γ_n^2 in [22] and the observation that on bounded subsets of L_0^2 the spectral gaps are uniformly bounded away from each other.

So we in particular have

$$n(x_n^2 + y_n^2) \sim nI_n \sim \gamma_n^2$$

locally uniformly on W. But while this gives us control of $x_n^2 + y_n^2$ in terms of γ_n^2 on the *real* space L_0^2, where all quantities are real, it does *not* so on the *complex* neighbourhood W, which we need to consider as well to establish analyticity properties. Indeed, for a non-real potential u, a gap length γ_n and thus an action I_n may vanish, while the Birkhoff coordinates x_n, y_n do *not*.

Additional data are necessary in this case. These are provided by the difference of the *Dirichlet eigenvalues* of L_u on $[0, 1]$ and the *spectral midpoints* of u,

$$\delta_n(u) = \mu_n(u) - \frac{\lambda_{2n-1}(u) + \lambda_{2n}(u)}{2}, \qquad n \geqslant 1.$$

They are real analytic functions of u on some complex neighbourhood of L_0^2. The quantities

$$\Delta_n(u) = |\gamma_n(u)| + |\delta_n(u)|$$

then take the role of the gap length for complex potentials – see [14, 21]. For real potentials, one has $0 \leqslant \delta_n \leqslant \gamma_n$ and hence $\gamma_n \sim \Delta_n$, since $\mu_n \in [\lambda_{2n-1}, \lambda_{2n}]$.

Proposition 2 ([22, p. 85]) *There exists a complex neighbourhood W of L_0^2 such that*

$$n\left(|x_n^2(u)| + |y_n^2(u)|\right) = O(\Delta_n^2(u)), \qquad n \geqslant 1,$$

locally uniformly on W and uniformly on bounded subsets of L_0^2.

The asymptotic behavior of the $\Delta_n(u)$ is intimately connected with the regularity of the potential u.

Proposition 3 ([21, 31]) *Let w be a submultiplicative weight. If $u \in \mathcal{H}_{0,\mathbb{C}}^w$, then*

$$\sum_{n\geqslant 1} w_n^2 \Delta_n^2(u) < \infty.$$

Moreover, the sum is locally uniformly bounded on $\mathcal{H}_{0,\mathbb{C}}^w$ and uniformly bounded on bounded subsets of $\mathcal{H}_0^w$.

We now combine the last two propositions to show that Ω maps $\mathcal{H}_0^w$ *into* $h_\star^w$ for all weights under consideration.

Corollary 4 *Let w be any submultiplicative weight. Then Ω is a bounded map from $\mathcal{H}_0^w$ into $h_\star^w$.*

Proof. By Propositions 2 and 3 we have

$$\sum_{n\geqslant 1} n w_n^2 \left(|x_n^2(u)| + |y_n^2(u)|\right) \leqslant c \sum_{n\geqslant 1} w_n^2 \Delta_n^2(u) < \infty$$

uniformly in some complex neighbourhood in $\mathcal{H}^w_{0,\mathbb{C}}$ around any potential $u \in \mathcal{H}^w_0$, and uniformly on bounded subset of L^2_0. Hence, Ω maps this neighbourhood into $\hbar^w_\star$ and is bounded. Moreover, as each coordinate function is real analytic and the map as a whole is locally uniformly bounded, Ω is also real analytic as a map from $\mathcal{H}^w_0$ into $\hbar^w_\star$ – see [22, Theorem A.5] or [32, Theorem A.3]. ∎

The second step is to show that the map of Corollary 4 is also *onto*. This is a consequence of Proposition 1 with a converse of Proposition 3, which, however, only applies to *subexponential* weights.

Proposition 5 ([13, 31]) *Let w be a subexponential weight. If $u \in L^2_0$ is such that*

$$\sum_{n\geqslant 1} w_n^2 \, |\gamma_n^2(u)| < \infty,$$

then $u \in \mathcal{H}^w_0$.

Corollary 6 *Let w be any subexponential weight. Then Ω maps the space $\mathcal{H}^w_0$ onto $\hbar^w_\star$.*

Proof of Corollary 6. Let $(x, y) \in \hbar^w_\star$. As Ω maps L^2_0 diffeomorphically onto the superset $\hbar^0_\star$ of $\hbar^w_\star$, there is a $u \in L^2_0$ with

$$\Omega(u) = (x, y).$$

By Proposition 1,

$$|\gamma_n^2(u)| \leqslant cn \, |I_n(u)| \leqslant cn \left(|x_n^2(u)| + |y_n^2(u)|\right), \qquad n \geq 1,$$

locally uniformly on a complex neighbourhood of L^2_0. Since $(x, y) \in \hbar^w_\star$,

$$\sum_{n\geqslant 1} w_n^2 \, |\gamma_n^2(u)| \leqslant c \sum_{n\geqslant 1} n w_n^2 \left(|x_n^2(u)| + |y_n^2(u)|\right) < \infty.$$

Using Proposition 5 it follows that u indeed belongs to $\mathcal{H}^w_0$. Thus, Ω is onto. ∎

Corollaries 4 and 6 together prove that Ω is a diffeomorphism between $\mathcal{H}^w_0$ and $\hbar^w_\star$ whenever w is a subexponential weight. Thus, Addendum 1 is proven.

Proposition 5 does not apply to exponential weights. This is exemplified by finite gap potentials such as the Weierstrass $\wp$-function, which are not entire functions. In this case, fixing any $r > 0$, we have

$$\gamma(u) \in \hbar^{r,a}_\star$$

for *all* $a > 0$, but *not* $u \in \mathcal{H}^{r,a}_0$ for all $a > 0$, as u has poles. Gasymov [16] even observed that *any* complex potential of the form

$$u = \sum_{n\geqslant 1} u_n \mathrm{e}^{2\pi i n x} = \sum_{n\geqslant 1} u_n z^n \Big|_{z=\mathrm{e}^{2\pi i x}}$$

is a 0-gap-potential. So in the complex case, the gap sequence need not contain *any* information about the regularity of the potential. – In the real case, however, we have the following classical result by Trubowitz. The very last statement is proven in [31].

Proposition 7 ([37]) *Let w be an exponential weight. If $u \in L^2_0$ is such that*

$$\sum_{n\geqslant 1} w_n^2 \, |\gamma_n^2(u)| < \infty,$$

then u is real analytic. More precisely, $u \in \mathcal{H}^{w^\rho}_0$ for some $0 < \rho \leqslant 1$, which depends only on the L^2-norm of u and a bound on the above sum.

Corollary 8 *If w is an exponential weight, then for any bounded subset B of $\hbar^w_\star$ there exists $0 < \rho \leqslant 1$ so that $\Omega^{-1}(B)$ is a bounded subset of $\mathcal{H}^{w^\rho}_0$.*

Proof. Let $A = \Omega^{-1}(B)$. As B is bounded in $\hbar^0_\star$ as well, A is bounded in L^2_0. By Proposition 1,

$$\sum_{n\geqslant 1} w_n^2 \, |\gamma_n^2(u)| \leqslant c \sum_{n\geqslant 1} n w_n^2 \left(|x_n^2(u)| + |y_n^2(u)|\right)$$

uniformly on A. The latter sum is uniformly bounded by

$$\sup_{(x,y)\in B} \|(x, y)\|^2_{\hbar^w_\star} < \infty,$$

since B is assumed to be bounded in $\hbar^w_\star$. It follows with Proposition 7 that there exists $0 < \rho \leqslant 1$ so that $A \subset \mathcal{H}^{w^\rho}_0$. Moreover, A is bounded in this space again by Proposition 3. ∎

4 Perturbations of KdV

Consider the perturbed KdV equation on some standard Sobolev space $\mathcal{H}_0^m$, $m \geqslant 1$, with

$$\frac{\partial u}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial H_c}{\partial u} + \varepsilon \frac{\partial K}{\partial u}\right) \tag{5}$$

for some real c, where H_c is given by (3). If K is real analytic in u with a gradient $\partial K/\partial u$ in $\mathcal{H}_0^m$, then KAM theory for KdV asserts the persistence of *quasi-periodic* solutions for sufficiently small $\varepsilon \neq 0$.

More precisely, given any $c \in \mathbb{R}$, any *finite* index set $A \subset \mathbb{N}$, and a compact subset $\Gamma \subset \mathbb{R}_+^A$ of positive Lebesgue measure, the majority of quasi-periodic solutions of (5) in $\mathcal{H}_0^w$ within

$$\mathcal{T}_\Gamma = \bigcup_{I \in \Gamma} \Omega^{-1}(T_I) \subset \mathcal{H}_0^m, \qquad T_I = \prod_{n \in A} S_{I_n} \times \{0\} \subset h_\star^m,$$

persists for sufficiently small $\varepsilon \neq 0$, being only slightly deformed. Consequently, the corresponding initial conditions lead to quasi-periodic solutions that stay in $\mathcal{H}_0^m$ for all time. This result may now be extended to weighted Sobolev spaces as follows.

Theorem 4* *Let $A \subset \mathbb{N}$ be a finite index set and $\Gamma \subset \mathbb{R}_+^A$ a compact subset of positive Lebesgue measure. Let w be a subexponential weight such that $\mathcal{H}_0^w \subset \mathcal{H}_0^1$, and let $c \in \mathbb{R}$. Assume that the Hamiltonian K is real analytic in a complex neighbourhood U of $\mathcal{T}_\Gamma$ in $\mathcal{H}_{0,\mathbb{C}}^w$ and satisfies the regularity condition*

$$\frac{\partial K}{\partial u} : \ U \to \mathcal{H}_{0,\mathbb{C}}^w, \qquad \sup_{u \in U} \left\| \frac{\partial K}{\partial u} \right\|_w \leqslant 1.$$

Then there exists $\varepsilon_0 > 0$ depending only on A, w, c and the size of U such that for $|\varepsilon| < \varepsilon_0$ the following holds. There exist

(i) *a nonempty Cantor set $\Gamma_\varepsilon \subset \Gamma$ with $\operatorname{meas}(\Gamma - \Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$,*

(ii) *a Lipschitz family of real analytic torus embeddings*

$$\Xi : \ \mathbb{T}^A \times \Gamma_\varepsilon \to U \cap \mathcal{H}_0^w,$$

(iii) *a Lipschitz map $\omega : \ \Gamma_\varepsilon \to \mathbb{R}^A$,*

such that for each $(\theta, I) \in \mathbb{T}^A \times \Gamma_\varepsilon$, the curve $u(t) = \Xi(\theta + \omega(I)t, I)$ is a quasi-periodic solution of (5) winding around the invariant torus $\Xi(\mathbb{T}^A \times \{I\})$. Moreover, each such torus is linearly stable.

A similar statement holds for exponential weights w, the only difference being that here,

$$\Xi : \ \mathbb{T}^A \times \Gamma_\varepsilon \to U \cap \mathcal{H}_0^{w^\rho},$$

with $0 < \rho \leqslant 1$ as in Proposition 7.

The proof of Theorem 4 is the same as in [22]. Essentially, one only has to replace the explicit weights e^{an} by the more general weights w_n.

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