Revised Version, July 2001
Ergod. Th. & Dynam. Syst. **22** (2002) 1537—1549

On the Construction of Almost Periodic Solutions for a Nonlinear Schrödinger Equation

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Dedicated to the memory of Jürgen Moser

1 Description of the Main Result

In this note we describe the construction of *almost-periodic solutions* for a nonlinear Schrödinger equation on a finite x-interval. The emphasis here will be on the method and not so much on the final result. Hence we will omit detailed proofs.

So far, there have been two types of results concerning the existence of almost-periodic solutions in infinite dimensional Hamiltonian systems. The first type concerns models in mathematical physics, which, roughly speaking, consist of lattices of harmonic oscillators with independent, identically distributed random frequencies with bounded or unbounded range, which are subject to anharmonic coupling forces, which are of finite range, short range, or hierarchical nature. Such infinite dimensional Hamiltonian systems are well approximable by finite dimensional ones, and one can show that for sufficiently small amplitudes of the oscillations, the majority of frequency distributions gives rise to almost-periodic motions of the entire ensemble of oscillators. That is, the evolution of the entire system in the appropriate phase space describes an almost-periodic curve with the maximal number of independent frequencies. See [P1] for a unified approach to these results and the references therein.

The second type of results concerns classical Hamiltonians of the form "kinetic energy + potential energy" in infinitely many action angle coordinates in the so called isochronous case, where the frequencies equal the actions: $\omega_n = I_n$ for $n = 1, 2, \ldots,$ while the potential is of general form. However, to find almost-periodic solutions one has to consider motions, whose frequencies ω_n tend to infinity *very fast*, namely, hyperexponentially in n. The upshot is that for such frequencies the classical diophantine estimates holds, and consequently the classical, finite dimensional KAM proof also works in this case, if one chooses proper norms. Such results were obtained in [CP] by a different method, which is in fact somewhat similar to the one described below.

Neither of these results applies to the nonlinear Schrödinger equation considered here. On one hand, the perturbations are not sufficiently localized to allow a rapid approximation by finite dimensional systems. On the other hand, the frequencies grow only quadratically, which is far too slow for the second approach to work. Thus, a different approach is necessary, which is the subject of this note.

The nonlinear Schrödinger equation to be considered is

$$\mathrm{i}u_t = u_{xx} - V(x)u - N(u), \qquad 0 \le x \le \pi, \tag{1}$$

with Dirichlet boundary conditions

$$u(t,0) = 0 = u(t,\pi), \qquad -\infty < t < \infty, \tag{2}$$

depending on a potential

$$V \in L^2_+ \subset L^2([0,\pi]),$$

where L^2_+ is the open subset of all potentials in $L^2([0,\pi])$ with strictly positive Dirichlet eigenvalues. The nonlinearity we *would like* to consider is

$$N(u) = f(|u|^2)u \tag{3}$$

with a function f, which is real analytic in a neighbourhood of $0 \in \mathbb{C}$ and vanishes at zero. However, for reasons to be explained later, we have to built in some nonlocal smoothing. Thus, we actually consider the nonlinearity

$$N(u) = \Psi\big(f\big(|\Psi u|^2\big)\Psi u\big), \tag{4}$$

where $\Psi\colon u \mapsto \psi * u$ is a convolution operator with an *even* function ψ on $\mathbb{R}$, which is smoothing of order $\sigma > 0$. More precisely,

$$\begin{gathered} \Psi\colon\ H_0^s([0,\pi]) \to H_0^{s+\sigma}([0,\pi]), \\ \|\Psi u\|_{H^{s+\sigma}} \le c_s\,\|u\|_{H^s}\,, \end{gathered} \tag{5}$$

for all $0 \leq s \leq 1$ with $\sigma > \frac{1}{4}$, and the convolution is defined by first extending u to an odd, 2π-periodic function on $\mathbb{R}$.

The nonlinearity is chosen so that the resulting partial differential equation is still Hamiltonian. For example, as the phase space one may take the Sobolev space $H_0^1([0, \pi])$ with the inner product $\langle u, v \rangle = \operatorname{Re} \int_0^\pi u\bar{v}\, dx$. The Hamiltonian then is

$$H = \frac{1}{2}\langle Lu, u\rangle + \frac{1}{2}\int_0^\pi F\big(|\Psi u|^2\big)\, dx,$$

where $L = -d^2/dx^2 + V$ and $F = \int_0 f\, dz$. The equations of motion are

$$\dot{u} = \mathrm{i}\nabla H(u),$$

where the gradient of H is defined with respect to $\langle \cdot\,, \cdot \rangle$, and the dot indicates differentiation with respect to time.

We note that the Dirichlet problem on $[0, \pi]$ is equivalent to the periodic problem with period 2π within the space of all *odd* functions, that is, to the boundary conditions

$$u(t, x) = u(t, x + 2\pi), \qquad u(t, -x) = -u(t, x).$$

This is due to the fact that the space of all odd functions is invariant under the evolution of (1) & (4), when ψ is *even*. Thus, we may equally well think of periodic boundary conditions within the space of all odd functions on $\mathbb{R}$. This has also the advantage that the convolution operator Ψ is defined naturally, without the need to extend u. Incidentally, in the same way Neumann boundary conditions are equivalent to periodic boundary conditions in the space of all even functions.

Our aim is to find solutions u for this problem that are almost-periodic in time. In fact, the solutions constructed are analytic in t and admit a rapidly converging Fourier series expansion

$$u(t, x) = \sum_{k \in \mathbb{Z}_0^{\mathbb{N}}} U_k(x) e^{\mathrm{i}\langle k, \omega\rangle t}, \tag{6}$$

where $\omega = (\omega_1, \omega_2, \dots)$ is an infinite sequence of rationally independent frequencies, and $\mathbb{Z}_0^{\mathbb{N}}$ is the space of all integer sequences $k = (k_1, k_2, \dots)$ with only finitely many nonzero components. It will turn out that the frequencies asymptotically tend to the Dirichlet eigenvalues of the potential V.

To construct these solutions, we consider the potential V as a *parameter* in the infinite dimensional domain L_+^2 which can be adjusted. Also, we look at solutions with *small amplitudes*, so that the nonlinearity amounts to a small perturbation of the linear equation. The main result can then be formulated as follows.

Main Theorem *For "almost all" potentials $V \in L^2_+$, the nonlinear Schrödinger equation* (1) *with Dirichlet boundary conditions* (2) *and nonlinearity* (4) *admits uncountably many analytic, almost-periodic solutions u of the form* (6) *in every neighbourhood of $u \equiv 0$ in $H^1_0([0,\pi])$.*

Here, "almost all" potentials means that the complementary set of potentials in L^2_+ has measure zero with respect to a large class of probability measures on L^2_+ described at the end of section 4. There we will also indicate that in a certain sense the set of all almost-periodic solutions has density 1 at $u \equiv 0$ in $H^1_0([0,\pi])$.

Almost periodic solutions for "typical" potentials have also been constructed by Bourgain [B], using the direct Newton-type method of Craig & Wayne [CW]. Indeed, this approach allows him to handle Schrödinger and wave equations with *local* nonlinearities, such as

$$u_{tt} = u_{xx} - V(x)u - \varepsilon f(u),$$

where f is an odd polynomial in u with $f(u) = O(|u|^3)$.

2 Reformulation of the Setting

We rewrite the nonlinear Schrödinger equation as a Hamiltonian system in infinitely many coordinates. To this end we make the ansatz

$$u(t,x) = \sum_{j \geq 1} q_j(t)\phi_j(x,V), \tag{7}$$

where $\phi_j(\,\cdot\,,V)$, $j = 1,2,\ldots$, are the normalized Dirichlet eigenfunctions of the potential V with Dirichlet eigenvalues $\lambda_j(V)$. Their asymptotic behaviour is

$$\lambda_j(V) = j^2 + [V] + \tilde{\lambda}_j(V), \qquad \tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots) \in \ell^2,$$

where $[V] = \frac{1}{\pi}\int_0^\pi V(x)\,dx$ denotes the mean value of V [PT].

The first step is to introduce the variable part of the frequencies, $\tilde{\lambda}$, as parameters instead of V. To simplify these matters we now restrict ourselves to the space

$$E = \left\{ V \in L^2_+ : [V] = 0,\ V(x) = V(\pi - x) \right\} \subset L^2_+$$

of all potentials in L^2_+, which have mean value zero and are even about $\frac{\pi}{2}$, and make use of the well known fact that the map

$$V \mapsto \tilde{\lambda}(V)$$

is a real analytic diffeomorphism between E and some open subset of ℓ^2 with a real analytic inverse [PT]. Thus, given any $V_0 \in E$, we have a real analytic diffeomorphism

$$B \to U, \quad \xi \mapsto V(\xi)$$

between an open ball $B \subset \ell^2$ and some open neighbourhood U of V_0 in E, such that $\lambda_j(V(\xi)) = j^2 + \xi_j$, hence

$$\tilde{\lambda}(\xi) \stackrel{\text{def}}{=} \tilde{\lambda}(V(\xi)) = \xi.$$

Moreover, the Dirichlet eigenfunctions ϕ_j also depend real analytically on ξ. From now on we fix such a ball $B \subset \ell^2$ of parameters ξ, since the Main Result is of a local nature with respect to the potential V.

We note that the mean value of $V \in L^2([0,\pi])$ only affects a uniform shift of the entire spectrum, while its odd part, very loosely speaking, only affects the shape of its Dirichlet eigenfunctions, not its eigenvalues – see [PT]. Hence, for our considerations these are unessential parameters, which we may ignore in the following.

Proceeding as in [KP] the ansatz (7) now leads to the ξ-dependent Hamiltonian

$$\begin{aligned} H &= \Lambda + G \\ &= \frac{1}{2} \sum_{j \geq 1} \lambda_j(\xi)\, |q_j|^2 + \frac{1}{2} \int_0^\pi F\big(|\Psi u(q,\xi)|^2\big)\, dx. \end{aligned}$$

We consider this system in a neighbourhood of zero in the phase space $\ell^2_{a,s}$ of all complex sequences $q = (q_1, q_2, \dots)$ with

$$\|q\|^2_{a,s} = \sum_{j \geq 1} |q_j|^2\, j^{2s} e^{2aj} < \infty,$$

where we may fix $a > 0$ and $s \geq 0$ arbitrarily. The symplectic structure on $\ell^2_{a,s}$ is $\frac{\mathrm{i}}{2} \sum_j dq_j \wedge d\bar{q}_j$. The Hamiltonian equations of motion are

$$\dot{q}_j = 2\mathrm{i} \frac{\partial H}{\partial \bar{q}_j}, \qquad j \geq 1. \tag{8}$$

This Hamiltonian may now serve as our new starting point due to the observation that every analytic solution curve $t \mapsto q(t)$ of (8) in $\ell^2_{a,s}$ for the parameter ξ defines through (7) an analytic solution of (1) & (2) for the potential $V(\xi)$. The proof is exactly the same as in [KP].

To proceed we formulate the essential properties of the new Hamiltonian, which are relevant for our argument.

Property A. The frequencies of the linear system have the asymptotic behavior

$$\lambda_j = j^2 + \tilde{\lambda}_j(\xi), \qquad \tilde{\lambda}_j(\xi) = O(1)$$

uniformly on B. Indeed, every other exponent $d > 1$ instead of 2 would do.

Property B. The map $\xi \mapsto \tilde{\lambda}(\xi)$ is a Lipschitz continuous map between some closed set $B \in \ell^2$ and another closed set in ℓ^2 with a Lipschitz continuous inverse. Indeed, in the present case, it is just the identity.

Property C. The sequence of partial derivatives G_q of G, which determines the nonlinear part of the Hamiltonian vector field of H, defines near zero a map of the type

$$\ell^2_{a,s} \to \ell^2_{a,s+2\sigma}, \quad q \mapsto G_q(q, \bar{q}; \xi)$$

for some $s \geq 0$, which is analytic in $q, \bar{q}$, Lipschitz in ξ, and satisfies

$$\left\| G_q \right\|_{a,s+2\sigma} = O\left(\|q\|^3_{a,s} \right). \tag{9}$$

Similarly for its Lipschitz semi-norm with respect to ξ, defined as in (13). That is, G_q is *smoothing* of order 2σ, where σ is the smoothing order of the convolution operator Ψ in (5).

The proof of these statements is completely parallel to the corresponding proof in [KP]. We point out that the assumption

$$2\sigma > \frac{1}{2}$$

is crucial in order to preserve the Lipschitz property B during the iterative process described below. For this reason we are not able to handle the case of local nonlinearities (3), for which $\sigma = 0$. It also does not help to let the mollifier ψ tend to the Dirac delta-function, since then the domain of validity of the Main Result shrinks to the point $u \equiv 0$.

3 Construction of Quasi-Periodic Solutions

The almost-periodic solutions are constructed by iterating a KAM theorem concerning the existence of quasi-periodic solutions. At each step, more oscillators are excited in order to increase the number of frequencies of the quasi-periodic solution. In the limit all oscillators are excited, and the resulting solution is almost-periodic with the maximal number of frequencies. In this section we are going to describe the first step of this scheme in detail. Its iteration is then indicated in the next section.

Fix an arbitrary finite integer $n \geq 1$, and introduce angle action coordinates $(x, y) \in \mathbb{T}^n \times \mathbb{R}^n$ for the first n oscillators by setting

$$q_j = \sqrt{2(I_j + y_j)}\, e^{\mathrm{i}x_j}, \qquad 1 \leq j \leq n. \tag{10}$$

Here, $I = (I_1, \dots, I_n)$ are the positive amplitudes of the first n oscillators which perform a quasi-periodic motion under the quadratic Hamiltonian Λ on the torus

$$\mathbb{T}^n(I) = \left\{ |q_j|^2 = 2I_j \text{ for } 1 \leq j \leq n \right\}.$$

To continue their motion for the full Hamiltonian $\Lambda + G$, we choose

$$I \in A_r = \left\{ r^2 < I_j < r^{2\beta} \text{ for } 1 \leq j \leq n \right\} \subset \mathbb{R}^n,$$

where $r > 0$ is chosen later and $\frac{2}{3} < \beta < 1$. Our aim is to continue these tori for a subset of amplitudes $I \in A_r$ whose relative measure tends to 1 as r tends to zero. However, in order to avoid technicalities we focus attention on *one* torus $\mathbb{T}^n(I)$ at a time in the following.

Up to terms which depend only on parameters and are dynamically irrelevant, we obtain in the new coordinates the Hamiltonian

$$H = \sum_{1 \leq j \leq n} \lambda_j(\xi) y_j + \frac{1}{2} \sum_{j \geq n+1} \lambda_j(\xi)\, |q_j|^2 + G(x, y, q', \bar{q}'; \xi, I), \tag{11}$$

where $q' = (q_{n+1}, q_{n+2}, \dots)$. On the complex neighbourhood

$$D(w, r) = \left\{ |\mathrm{Im}\, x| < w \right\} \times \left\{ |y| < r^2 \right\} \times \left\{ \|q'\|_{a,s} < r \right\}$$

of the torus $\mathbb{T}^n \times \{0\} \times \{0\}$ in the complexification of the phase space $\mathbb{T}^n \times \mathbb{R}^n \times \ell^2_{a,s}$, we have in view of (9)

$$\begin{aligned} |G|^{\sup}_{D(w,r)\times B\times A_r} &\stackrel{\text{def}}{=} \sup_{D(w,r)\times B\times A_r} |G| = O(r^{4\beta}), \\ \|G_q\|^{\sup}_{a,s+2\sigma;D(w,r)\times B\times A_r} &\stackrel{\text{def}}{=} \sup_{D(w,r)\times B\times A_r} \|G_q\|_{a,s+2\sigma} = O(r^{3\beta}). \end{aligned} \tag{12}$$

The same holds for their Lipschitz semi-norms with respect to ξ. For example,

$$|G|^{\text{lip}}_{D(w,r)\times B\times A_r} \stackrel{\text{def}}{=} \sup_{\xi\neq\zeta,\,\xi,\zeta\in B} \frac{\left|\Delta_{\xi\zeta} G\right|^{\sup}_{D(w,r)\times A_r}}{\|\xi-\zeta\|_{\ell^2}} = O(r^{4\beta}), \tag{13}$$

where $\Delta_{\xi\zeta} G = G(\,\cdot\,;\xi) - G(\,\cdot\,;\zeta)$.

In order to apply the KAM theory at this stage it is not necessary to have all the infinitely many parameters ξ at our disposal. It suffices to split them up as

$$\xi = (\xi^n, \xi') \in B^n \times B' \subset B,$$

where B^n is a closed ball in $\mathbb{R}^n$ and B' some closed ball of sequences $\xi' = (\xi_{n+1}, \xi_{n+2}, \ldots)$ in ℓ^2. At the moment the ξ' may be considered as dummy parameters with respect to which everything holds uniformly. It is the ξ^n which have to be chosen properly.

We recall from [KP, P2] the basic KAM theorem about the persistence of quasi-periodic motions in infinite dimensional Hamiltonian systems of the form (11), but now depending on an n-dimensional parameter $\zeta \in \Pi \subset \mathbb{R}^n$ instead of ξ. The aim is to show the persistence of the torus $\mathbb{T}^n \times \{0\} \times \{0\}$, which is invariant for $G = 0$, under the effect of a small nonlinearity $G \neq 0$. To this end the following assumptions are made.

Assumption A: Spectral Asymptotics. The frequencies satisfy

$$\lambda_j(\zeta) = j^d + \ldots + \tilde{\lambda}(\zeta), \qquad \tilde{\lambda}_j(\zeta) = O(j^\delta),$$

with some $d > 1$ and $\delta < d-1$, where the dots stand for fixed, ζ-independent lower order terms. Moreover,

$$|\tilde{\lambda}|^{\text{lip}}_{-\delta,\Pi} \leq M,$$

where $|\tilde{\lambda}|_{-\delta} = \sup_j j^{-\delta}\,|\tilde{\lambda}_j|$.

We remark that the case $d = 1$ may also be considered. We omit it here, since the statement is more involved. See [P2] for the details.

Assumption B: Nondegeneracy. The map $\omega\colon \zeta \mapsto (\lambda_1(\zeta), \ldots, \lambda_n(\zeta))$ is a one-to-one map between the closed set Π and its image in $\mathbb{R}^n$, which is Lipschitz continuous in both directions, with

$$\left|\omega^{-1}\right|^{\text{lip}}_{\omega(\Pi)} \le L < \infty. \tag{14}$$

Moreover,

$$\operatorname{meas}\left\{\zeta \in \Pi : \langle k, \lambda(\zeta)\rangle = 0\right\} = 0$$

and

$$\langle k', \lambda'(\zeta)\rangle \ne 0 \quad \text{on } \Pi$$

for all nonzero integer vectors $k \in \mathbb{Z}_0^{\mathbb{N}}$ with $|k'| = \sum_{j \ge n+1} |k_j| \le 2$ and $\lambda' = (\lambda_{n+1}, \lambda_{n+2}, \ldots)$. Here, 'meas' denotes Lebesgue measure.

Assumption C: Regularity. The higher order term G is analytic in the space coordinates $(x, y, q, \bar{q})$ and Lipschitz in the parameter ζ on the domain $D(w, r) \times \Pi$ for some positive w and r. Moreover, its q-derivative defines a likewise regular map into $\ell^2_{a,s+\rho}$ for some $\rho \ge 0$, such that also $\rho \ge -\delta$.

We note that for $\rho > 0$ the map G_q is *smoothing* of order ρ. This property is not necessary for the KAM theorem to apply, but it does affect the estimates for the frequencies λ^+, which is important for our purposes. Also note that we may always increase δ to meet the condition $\rho \ge -\delta$.

The following KAM theorem is proven in [K1] and [P2].

The Basic KAM Theorem *Suppose the Hamiltonian H in* (11) *satisfies assumptions A, B and C, and that $\delta \le 0$. Then there exists a constant $\gamma > 0$ depending only on n, d, w, the product LM and another parameter $0 < \theta < 1$ such that the following holds.*

If, on $D(w, r) \times \Pi$,

$$\begin{aligned}\varepsilon = \frac{1}{r^2}\left(|G|^{\text{sup}} + \frac{\alpha}{M}|G|^{\text{lip}}\right)& \\ + \frac{1}{r}\left(\|G_q\|^{\text{sup}}_{a,s+\rho} + \frac{\alpha}{M}\|G_q\|^{\text{lip}}_{a,s+\rho}\right) &\le \alpha\gamma\end{aligned} \tag{15}$$

for some $0 < \alpha \le 1$, then there exists

(i) *a Cantor set $\Pi_\alpha \subset \Pi$ of parameters with* $\operatorname{meas}(\Pi - \Pi_\alpha) \to 0$ *as $\alpha \to 0$, in particular*

$$\operatorname{meas}(\Pi - \Pi_\alpha) = O(\varrho^{n-1}\alpha), \qquad \varrho = \operatorname{diam}\Pi,$$

if the frequencies λ are affine functions of ξ;

(ii) *a new frequency map* $\zeta \mapsto \lambda^+(\zeta)$ *on* Π_α *with*

$$\left|\lambda^+ - \lambda\right|^{\sup}_{\rho,\Pi_\alpha} + \frac{\alpha}{M}\left|\lambda^+ - \lambda\right|^{\text{lip}}_{\rho,\Pi_\alpha} \leq c\varepsilon,$$

where c *depends on the same parameters as* γ*; and*

(iii) *a family of symplectic coordinate transformations*

$$\Phi\colon D(\theta w, \theta r) \times \Pi_\alpha \to D(w, r),$$

which is close to the identity, analytic in $x, y, q, \bar{q}$ *and Lipschitz in* ζ*;*

such that

$$H \circ \Phi = \sum_{1 \leq j \leq n} \lambda_j^+(\zeta) y_j + \frac{1}{2} \sum_{j \geq n+1} \lambda_j^+(\zeta)\,|q_j|^2 + G^+(x, y, q, \bar{q}; \zeta),$$

where G^+ *is of the same regularity as* G *and of order three in* $y, q, \bar{q}$*. That is, its Taylor series expansion contains only monomials* $y^l q^m \bar{q}^{\bar{m}}$ *with* $2\,|l| + |m + \bar{m}| \geq 3$*.*

Consequently, for each $\zeta \in \Pi_\alpha$ *the Hamiltonian* H *admits an invariant torus with frequencies* $\lambda_1^+(\zeta), \ldots, \lambda_n^+(\zeta)$*, which is the image of* $\mathbb{T}^n \times \{0\} \times \{0\}$ *under* $\Phi(\,\cdot\,; \zeta)$*.*

Remark 1. The parameter set Π may be any closed subset of $\mathbb{R}^n$. In particular, it may be a Cantor set. The constant γ is independent of Π.

Remark 2. The parameter θ controls the domain of analyticity of Φ in (iii) and hence of the transformed Hamiltonian $H \circ \Phi$, and $\gamma \to 0$ as $\theta \to 1$. The role of the parameter α will become clear in a moment.

Remark 3. We forego the detailed estimates in which sense Φ is close to the identity. Suffice it to say that a weighted sup-norm of $\Phi - id$ plus α/M times the corresponding Lipschitz semi-norm is of the order of ε/α.

Remark 4. In the case $0 < \delta < d - 1$ the result is the same, but the constant γ has to depend on more detailed asymptotics of the frequencies for (i) to be correct.

Remark 5. Assumption C and in particular the assumption on ρ can be relaxed to $\rho > -(d-1)$. Thus, as an operator the nonlinearity G_q may even be unbounded of an order less than $d - 1$. This version of the KAM theorem is for example applicable to Hamiltonian perturbations of KdV equations. Its proof, however, requires considerably more refined estimates. It can be found in [K2] and [KPI].

For our purpose it is important to note that additional parameters may be included as follows. Again write $\xi = (\xi^n, \xi') \in B^n \times B' = B$, and suppose that assumptions A, B, C hold with respect to $\xi^n \in B^n$ *uniformly* with respect to $\xi' \in B'$, including Lipschitz estimates with respect to *all* ξ. In particular, assume that

$$|\tilde{\lambda}|^{\mathrm{lip}}_{-\delta,B} \le M, \qquad |\omega^{-1}|^{\mathrm{lip}}_{\omega(B^n)\times B'} \le L,$$

where ω^{-1} is now a function of ω and ξ'.

Then the conclusions hold uniformly with respect to ξ', including Lipschitz estimates. For example, the Cantor set $\Pi_\alpha \subset B^n$ now depends on ξ' in a Lipschitz fashion, and we have

$$\left|\lambda^+ - \lambda\right|^{\sup}_{\rho,B_+} + \frac{\alpha}{M}\left|\lambda^+ - \lambda\right|^{\mathrm{lip}}_{\rho,B_+} \le c\varepsilon \tag{16}$$

uniformly on the domain $B_+ = \{(\xi^n, \xi') : \xi^n \in \Pi_\alpha(\xi')\} \subset B$. Likewise, the measure estimates hold uniformly with respect to $\xi' \in B'$, as do the Lipschitz properties of Φ. These statements are verified by inspecting the proof in [P2].

The assumptions of the Basic KAM Theorem are now easily checked. We have $\lambda_j(\xi) = j^2 + \xi_j$ and $\tilde{\lambda}_j(\xi) = \xi_j$, so assumptions A and B are clearly satisfied with $d = 2$, $\delta = 0$, and

$$|\tilde{\lambda}|^{\mathrm{lip}}_{0,\omega(B^n)\times B'} = 1, \qquad \left|\omega^{-1}\right|^{\mathrm{lip}}_{\omega(B^n)\times B'} = 1$$

for any $n \ge 1$. Also, assumption C is satisfied for some fixed w, say $w = 1$, all sufficiently small $r > 0$, and $s \ge 0$, $\rho = 2\sigma > \frac{1}{2}$. Finally, for $0 < \alpha < 1$, we uniformly have in (15)

$$\varepsilon = O(r^{4\beta-2})$$

in view of (12). So we may choose for example

$$\alpha = r^{\beta/4}$$

and $\frac{2}{3} < \beta < 1$ to satisfy the smallness condition for any fixed $0 < \theta < \frac{1}{2}$ and all sufficiently small $r > 0$. In addition, we also have $\varepsilon/\alpha^2 \to 0$ as $r \to 0$, which is required in order to control all estimates mentioned in Remark 3 above.

As a result we obtain a new Cantor-like set $B_+ \subset B$ of the form

$$B_+ = \left\{(\xi^n, \xi') : \xi^n \in \Pi_\alpha(\xi')\right\}$$

with

$$\frac{\mathrm{meas}(B^n - \Pi_\alpha(\xi'))}{\mathrm{meas}(B^n)} = O(\alpha)$$

uniformly in ξ', and a Lipschitz family of symplectic coordinate transformations Φ: $D(\theta w, \theta r) \times B_+ \to D(w, r)$ close to the identity such that

$$H \circ \Phi = \sum_{1 \le j \le n} \lambda_j^+(\xi) y_j + \frac{1}{2} \sum_{j \ge n+1} \lambda_j^+(\xi)\, |q_j|^2 + O_3(y, q', \bar{q}') \tag{17}$$

with new frequencies λ^+ satisfying (16) and higher order terms of the same regularity as in assumption C. Hence, the given Hamiltonian H admits an embedded invariant torus close to $\mathbb{T}^n \times \{0\} \times \{0\}$ for each parameter $\xi \in B_+$ carrying quasi-periodic motions with frequencies $\lambda_1^+(\xi), \ldots, \lambda_n^+(\xi)$.

It is worth pointing out that these tori are also embedded tori in the rectangular coordinates q, because their deviation in the y-direction from $y = 0$ is much smaller than r^2 due to the estimates of Φ not given here. Consequently, in (10) we have

$$\tfrac{1}{2}\, |q_j|^2 = I_j + y_j \ge \tfrac{1}{2} r^2$$

everywhere for $I \in A_r$.

4 Iteration of the KAM Theorem

The "output" of the Basic KAM Theorem, the transformed Hamiltonian $H \circ \Phi$ on $D(\theta w, \theta r) \times B_+$, is almost in a form suitable to serve again as "input" for the construction of the preceding section. What is missing yet is a control of the Lipschitz properties of the new frequencies λ^+, in particular (14), in a uniform way. It is at this point where the smoothing hypothesis

$$\rho = 2\sigma > \frac{1}{2}$$

is required.

The KAM theorem only provides an estimate of $\lambda^+ - \lambda$ in the weighted sup-norm $|\cdot|_\rho$. By the above assumption on ρ, however, this implies an estimate in the ℓ^2-norm $\|\cdot\|_{\ell^2}$, namely

$$\left\|\lambda^+ - \lambda\right\|_{\ell^2} \le c_\rho \left|\lambda^+ - \lambda\right|_\rho \le c\varepsilon$$

for $\rho > \frac{1}{2}$. The same applies to their Lipschitz estimates, giving

$$\left\|\lambda^+ - \lambda\right\|_{B_+}^{\mathrm{lip}} \le \frac{c\varepsilon}{\alpha} \le O(r^{3\beta-2}).$$

It follows that the new “frequency tail”

$$\tilde{\lambda}^{+} \colon B_{+} \to \ell^{2}, \quad \xi \mapsto \tilde{\lambda}^{+}(\xi)$$

is as Lipschitz close to $\tilde{\lambda} = id$ in ℓ^2 as we need by making r small. Then assumptions A and B can be met again with any choice of another n.

Another minor difference concerns the higher order terms of the Hamiltonian. Now they are of order three (in the sense of the KAM theorem) rather than of order four. But this causes no difficulties.

At the next step of the iterative construction, and likewise in all subsequent steps, we thus do the following. Take the Hamiltonian $H_+ = H \circ \Phi$ on $D(w_+, r_+) \times B_+$, where $w_+ = \theta w$ is fixed, and $r_+ \le \theta r$ will be made small. Choose any $n_+ > n$, and “open up” more oscillators by writing

$$q_j = \sqrt{2(I_j + y_j)}\, e^{\mathrm{i}x_j}, \qquad n < j \le n_+,$$

with

$$I^+ \in A^+_{r_+} = \left\{ r_+^2 < I_j < r_+^{2\beta} \text{ for } n < j \le n_+ \right\}. \tag{18}$$

Fixing any such I^+, we obtain

$$H_+ = \sum_{1 \le j \le n_+} \lambda_j^+(\xi) y_j + \frac{1}{2} \sum_{j \ge n_+ + 1} \lambda_j^+(\xi)\, |q_j|^2 + G_+(x, y, q'', \bar{q}''; \xi, I),$$

where with the correspondingly defined domain $D_+ = D_+(w_+, r_+)$,

$$\left|G^+\right|^{\sup}_{D_+ \times B_+} = O(r_+^{2\beta+1}), \qquad \left\|G_q^+\right\|^{\sup}_{a, s+\rho, D_+ \times B_+} = O(r_+^{2\beta}).$$

The same applies to their Lipschitz semi-norms. Hence, the smallness condition for the next application of the KAM theorem reduces to

$$\varepsilon_+ = O(r_+^{2\beta - 1}) \le \alpha_+ \gamma_+.$$

Thus we may choose again $\alpha_+ = r_+^{\beta/4}$ and $\frac{2}{3} < \beta < 1$ for all r_+ sufficiently small.

Finally, we again split up the parameters $\xi \in B_+$ in their “essential” and their “dummy” parts,

$$\xi = (\xi^{n_+}, \xi'').$$

It is, however, not necessary to write B_+ as a corresponding product of two closed sets. It suffices to consider the slices

$$B^{n_+}(\xi'') = \left\{ \xi^{n_+} : (\xi^{n_+}, \xi'') \in B_+ \right\},$$

since it is sufficient to have all estimates uniform with respect to ξ''.

The upshot is that after choosing the integer $n_+ > n$ and the next shrinking factor $0 < \theta_+ < 1$ arbitrarily, then determining all the constants involved such as Lipschitz semi-norms for the frequencies, we still have the freedom to choose r_+ as small as needed to meet the resulting smallness condition of the KAM theorem.

In this manner the KAM theorem may now be iterated infinitely often. At each step we are given a Hamiltonian of the form (17). We open up some more oscillators by introducing more action-angle-variables depending on parameters I as in (18) in some punctured neighbourhood of the origin. Making this neighbourhood sufficiently small the remaining perturbation can be made as small as needed to apply the KAM theorem.

We obtain an increasing sequence of integers $n_0 < n_1 < \ldots$, a decreasing sequence of Cantor-like parameter sets $B_0 \supset B_1 \supset \ldots$, and parameter dependent analytic embeddings Φ_ν, $\nu = 1, 2, \ldots$, of the tori $\mathbb{T}^{n_\nu}$ into the phase space of the original Hamiltonian H, filled with quasi-periodic motions with frequencies $\lambda_1^\nu, \ldots, \lambda_{n_\nu}^\nu$ close to the unperturbed frequencies $\lambda_1, \ldots, \lambda_{n_\nu}$.

The convergence of this scheme presents no difficulties, since the subsequent perturbations as well as the finite-dimensional measures of $B_\nu - B_{\nu+1}$ can be made as small as needed. Hence, in the limit, we obtain a Cantor set $B_* = \bigcap B_\nu$ of parameters ξ in ℓ^2, parametrizing analytic embeddings Φ_* of the infinite-dimensional torus $\mathbb{T}^{\mathbb{N}}$ into the original phase space and giving rise to almost-periodic motions with frequencies $\lambda_1^*, \lambda_2^*, \ldots$, converging asymptotically to the original frequencies for this parameter. Indeed, these embeddings are analytic in a uniform strip $\{ |\mathrm{Im}\, x_j| < w_*, j \geq 1 \}$ around $\mathbb{T}^{\mathbb{N}}$ and admit very rapidly converging Fourier series expansions. Also, all the frequencies are rationally independent.

We now verify that in a certain sense these solutions exist for almost all potentials $V \in E$. Since this is a local statement we may fix some potential V_0, some neighbourhood U in E of it, and consider some probability measure μ on L^2. Passing to the space of parameters ξ, we obtain a neighbourhood B around some ξ_0 with a corresponding probability measure η. We *assume* that the projection of this measure onto any affine subspace of ℓ^2 is absolutely continuous with respect to Lebesgue measure.

Now, at every step of the iterative construction we are excluding a parameter set

$$X = \left\{ (\xi^n, \xi') : \xi^n \in B^n(\xi') - \Pi_\alpha^n(\xi') \right\},$$

where

$$\text{meas}\left(B^n(\xi') - \Pi^n_\alpha(\xi')\right) = O(\alpha(r)) \to 0 \qquad \text{as } r \to 0,$$

and 'meas' denotes the n-dimensional Lebesgue measure. By our assumption and Fubini's theorem, we then also have

$$\eta(X) = O(\alpha(r)) \to 0 \qquad \text{as } r \to 0$$

at each step. It follows that the total measure of all excluded parameters can be made as small as we wish, just by looking at a sufficiently small neighbourhood of the zero solution. Consequently, our construction applies to almost all $\xi \in B$ and hence also to almost all potentials $V \in E$ in the appropriate sense. The extension of this statement to almost all $V \in L^2_+$ presents no new difficulties.

We finally indicate, why there are uncountably many almost-periodic solutions for almost all potentials, and in which sense they have density 1 at $u \equiv 0$.

Initially, the construction of these solutions depends on the successive choices of the amplitude parameters $I = (I_1, I_2, \dots)$ in some fixed parameter domain A_*, which is the product of the blocks A_r in (18). So does the large Cantor set B_* of ξ-parameters. That is, we have $B_* = B_*(I)$. It is not difficult to arrange that all these dependencies are Lipschitz close to the identity. One may then turn this dependence around and find another similarly large, but *fixed* Cantor set $B^* \subset B_*$ and Cantor sets $A^* = A^*(\xi)$ for each $\xi \in B^*$, such that for each (ξ, I) with $\xi \in B^*$, $I \in A^*(\xi)$, there is a corresponding almost-periodic solution. Hence, for these ξ one has uncountably many almost-periodic solutions.

One may then show that for almost all ξ, hence for almost all $V \in E$, the set $A^*(\xi)$ has density 1 at $0 \in \mathbb{R}^{\mathbb{N}}_+$ in the sense that its intersection with any subspace of finitely many amplitudes has density 1 at 0 with respect to Lebesgue measure. To see this for a given such subspace one just has to start the zeroth step of the iteration with a sufficiently large n to include this space in $\mathbb{R}^n$ and to make r arbitrarily small.

5 Some Remarks concerning Open Problems

Remark 1. The radii of the invariant tori, which are approximately the parameters I_n, $n \geq 1$, are all positive, but shrink to zero *very rapidly*. Hence these tori are all compact. Indeed, we can make these radii shrink as fast as we wish. On the other hand, they surely have to shrink more than exponentially with n due to the dependence of the KAM estimates on the dimension of the invariant tori. We did not investigate the question about how fast *at least* the radii have to shrink.

Remark 2. The smoothing operator Ψ had to be introduced into the nonlinearity (4) in order to keep the perturbations of the parameter-frequency map $\xi \mapsto \tilde{\lambda}(\xi)$ small in the ℓ^2-norm $\|\cdot\|_{\ell^2}$, and not only in the sup-norm $|\cdot|_0$, as would be the case for a local nonlinearity such as $f(|u|^2)u$. It is an interesting problem whether it is possible to remove this restriction when using KAM theory. Apparently, this problem does not arise in Bourgain's approach [B].

Remark 3. The problem is also greatly simplified by the assumption that some potential is available serving as an infinite dimensional parameter. This decouples the problem of choosing amplitudes for the action angle coordinates and of adjusting the frequencies. In particular, we are free to choose these amplitudes as small as we wish.

Nothing is known, however, about the existence of almost-periodic solutions for a nonlinear Schrödinger equation such as

$$\mathrm{i}u_t = u_{xx} - mu - f(|u|^2)u$$

with Dirichlet boundary conditions on $[0, \pi]$, although for example a complete, nondegenerate Birkhoff normal form up to order four is available [KP].

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